



Modeling Borehole Interactions and In-situ Leach Uranium Mining Impacts on Groundwater in the Stampriet Aquifer

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Abstract

This study investigates the implications of in-situ leach (ISL) mining for uranium in the Auob aquifer of the Stampriet Transboundary Aquifer System (STAS). A mathematical model accounting for complex real-world processes, including heterogeneity, borehole interactions, and the dynamics of the leaching process was derived to simulate groundwater abstraction and injection in a confined aquifer. Extensive model analyses were performed, including sensitivity analysis of storativity and transmissivity, contributions of individual boreholes, and asymptotic behaviour of the system in early and late time. Simulations were conducted, including the radius of influence, mass balance, and flow dynamics analysis considering the reinjection effect with different transfer efficiencies. An iterative approach was defined for implementation and examples are provided in terms of figures and maps of the decision-making process of evaluating if the ISL mining places an unacceptable burden on water quality and resource sustainability. The results underscore the risks of ISL activities for the STAS and provide approaches towards reducing possible contamination with a parallel focus on optimizing uranium recovery.

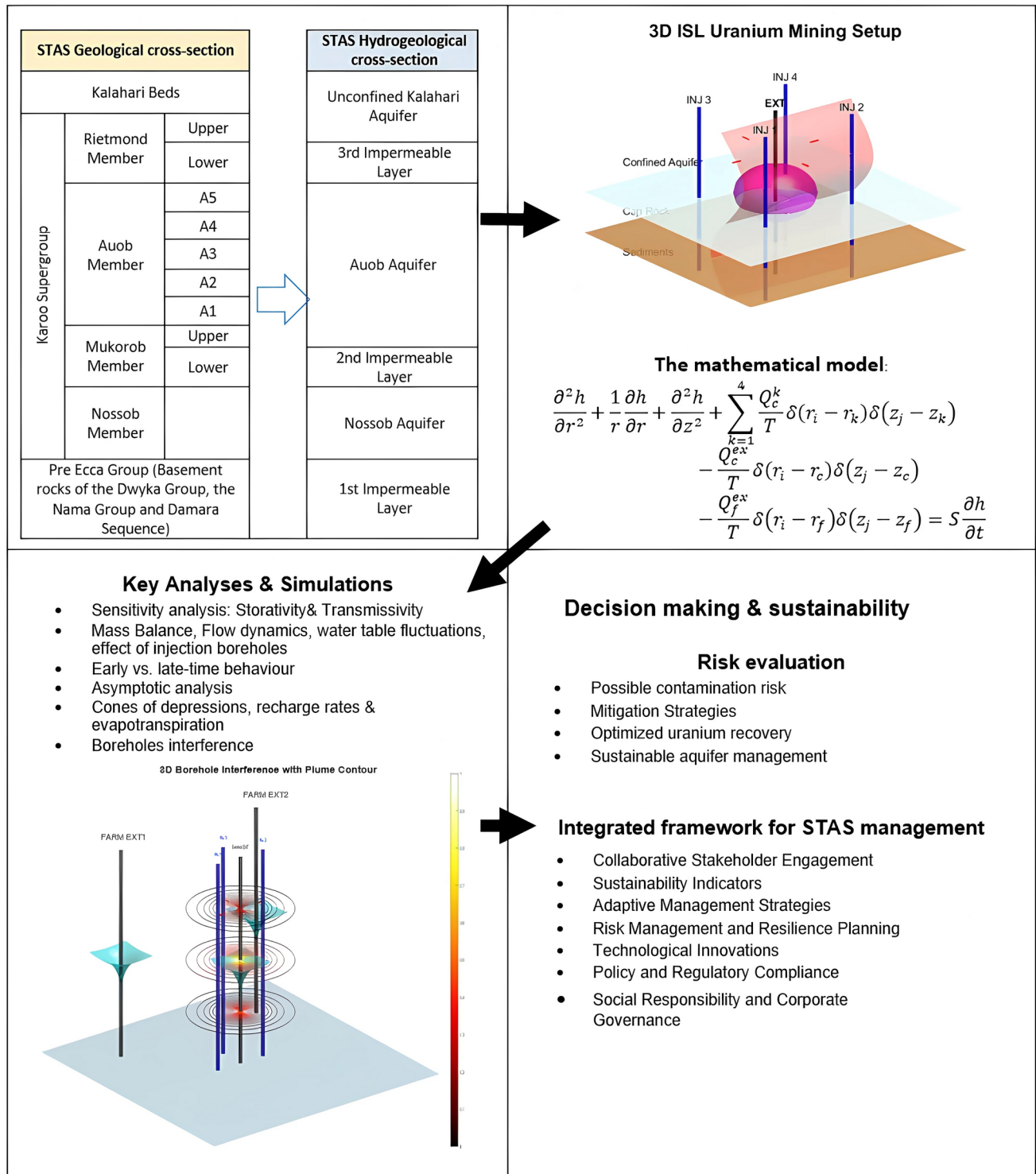
Graphical Abstract

The graphical abstract gives the overview of the article that focuses on in-situ leach (ISL) uranium mining in the Auob aquifer of the Stampriet Transboundary Aquifer System (STAS). It includes the study area with borehole localization for groundwater abstraction and reinjection, a sketch of the mathematical model used for leaching, and very detailed results including sensitivity studies of storativity and transmissivity. Furthermore, it refers to the cone of depression and radius of influence. Additionally, decision-making variables about potential contamination risks, mitigation measures, and maximization of uranium recovery are described in the abstract. The sections are visually broken down into clearly designated categories, enabling researchers to interpret complex relationships between the facets of the model and the wider aims of the research with respect to sustainability.

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1 Introduction

In-situ leaching (ISL) is a well-developed hydrometallurgical method which isolates uranium directly from ore in the subsurface, by injecting leaching reagents and recovering the parent solution through pumping wells. This process is generally applied in sandstone-type Uranium ores found in confined aquifers that are saturated (Karami et al. 2023; Saunders et al. 2016); where groundwater acts as both the transport conduit of the leaching solutions and the produced parent solution. In-situ leaching is a process where rock-ore reactions and the movement of leaching solutions interact, forming a leaching zone (Nguyen et al. 1983). This area is affected by hydrodynamics and chemical reaction kinetics, which allow for the contact and reaction of chemical reagents with in-situ ore (Tan et al. 2005). Various leaching solutions—including acid, alkaline or $\text{CO}_2\text{-O}_2$ solutions—are mixed with native groundwater and used to leach uranium from the ore. Once these uranium-bearing solutions are generated, it is mechanically pumped to the surface through a series of pumping wells carefully drilled into the aquifer (Li and Yao 2024; Yao et al. 2023; Yuan and Niu 2018).

An ISL process itself generally consists of a network of injection and abstraction wells arranged according to the geological and hydrogeological characteristics of the subsurface. This arrangement is typically based on a 5-spot or 7-spot well pattern to maximize uranium recovery, while minimizing environmental footprints (Li and Yao 2024). ISL is a commonly practiced in many large uranium producing countries (such as Kazakhstan, Uzbekistan, China, USA, Canada), hence approximately 57% of the world total uranium is produced via this mining method (WNA 2024). Although ISL mining is deemed efficient, its application in freshwater aquifers is quite problematic in terrestrial regions where groundwater is the only available and reliable water source for household, agricultural, and livestock needs. ISL technology is very attractive as it has the potential to minimize surface disturbance compared to traditional mining methods.

Nonetheless, to ensure the efficiency of this mining method, a deep cone of depression is produced in the aquifer via excess water abstraction during ISL mining, contributing to the potential hazards of ISL mining on the availability of water for all (Saunders et al. 2016). This approach can result in decreasing groundwater levels affecting other users who share the same aquifer as their water source (Liu et al. 2017; Zhang et al. 2022). Especially in arid regions where water availability is already stretched beyond limits because of diminished recharge rates and climate variability, the long-term effects of ISL mining are highly alarming (Zhang et al. 2021). These issues are also exacerbated by the

use of pumped solutions being introduced into the evaporation ponds, especially with low annual precipitation periods. In light of these concerns, there remains a critical gap in detailed mapping of ISL mining's impact on groundwater resources in the Stampriet Transboundary Aquifer System (STAS); a critical transboundary aquifer shared between Namibia, South Africa and Botswana. The larger portion of the STAS is located in Namibia, where groundwater levels are decreasing and climate change is impacting the basin; consequently the competition is rising among water users, including farmers, who depend heavily on the aquifer for irrigation and livestock watering (ORASECOM 2016). Due to the proposal to implement the ISL mining for uranium in the Auob aquifer, which is part of the STAS, it is imperative to understand the potential impacts on other water users in the area with boreholes in the close proximity of the mining area.

This study thus proposes to fill this gap by creating a robust mathematical model to simulate the hydrodynamic effects of ISL mining on the surrounding aquifer system concerning the boreholes interactions based on multiple boreholes cones of depression. This model will further assist us to understand in terms of the groundwater levels, water quality, and sustainability of water resources in the context of ISL mining in the area. We will analyse the sensitivity of the impacts of the ISL process on the aquifer by varying storativity, transmissivity, and borehole contributions to evaluate mitigation strategies to minimize impacts. This study highlights are highly important for ensuring that effective policies addressing both sustainable groundwater management and uranium extraction are put in place, especially in emerging water-limited regions where demand is rising but climate change has severely reduced availability. Hence, this paper presents an innovative mathematical framework to simulate the complicated processes involved, to highlight hydrological and environmental risks, and to offer specific recommendations for actions to implement in the region as a needed addition to the current ISL mining knowledge base. The model, with its comprehensive treatment of borehole performance and consideration of the induced cone of depression in the Auob aquifer, will provide a useful guidepost for evaluating the viability of ISL mining in the hydrogeological settings like the Stampriet Transboundary Aquifer System.

2 Materials and Methods

2.1 Study Area Description

The recently discovered sandstone-type uranium deposit is located in the Leonardville region, northwest of the

Stampriet Transboundary Aquifer System (STAS) or Stampriet Basin. Commonly called the STAS, the aquifer shared by South Africa, Namibia, and Botswana, is a vital transboundary aquifer supplying water for agriculture, tourism, and domestic uses in these countries (Kinoti et al. 2024). The Stampriet Basin comprises three main aquifers, namely the Kalahari, the Auob and the Nossob aquifer. These aquifers are the region's only reliable water source. This study is limited to a segment of the STAS which occurs in Namibia specifically the Auob aquifer. According to Mendelsohn et al. (2002), the region is semi-arid, with annual average rainfall between 150 and 300 mm which declines from north-east to southwest. The average annual temperature varies between 19 °C to 22 °C (Lohe et al. 2020). Hydrologically, the Nossob and Auob rivers are ephemeral, flowing only after occasional heavy downpours. Groundwater is the area's most prized resource, since there are no perennial rivers in the Stampriet Basin (Mapani et al. 2023).

Validation that in-situ leaching (ISL) was a feasible method to recover uranium from the sandstone-type deposit located within the Auob aquifer using sulphuric acid was confirmed through hydrogeological and geological exploration drilling, which was completed in 2021 (NEA 2023). The next phase of the project is an ISL pilot test. This step is crucial for generating hydrometallurgical data needed to assess the economic feasibility of ISL activities and to fulfil commercial licensing obligations (Ruiz et al. 2019). There has been considerable resistance though, from affected water users who worry that the aquifer could be polluted when sulphuric acid is injected throughout that process. Their concerns highlight the balancing act between resource exploitation and water resource protection, especially in such a water stressed area. At present, Namibia's governing body for the managing of water resources has not approved the ISL pilot testing. This decision is squarely within the public interest, as it is appropriate to protect critical groundwater reserves, as well as ensuring that extraction methods promote public trust in the safety and sustainability of resource extraction.

2.2 Methodology

We have to admit that there is currently a lack of site-specific data on the feasibility and safety of ISL in the Auob aquifer; as of now, no pilot testing has taken place. However, understanding the consequences of ISL activity is crucial for guiding sound regulation, so the absence of data poses a serious challenge for regulators. Here, we use a hypothetical analysis to demonstrate how ISL mining could potentially generate a deep cone of depression, taking into account the implications for nearby water users.

2.2.1 Assumptions and Scope

For the purposes of this analysis, we posit:

- i. The proposed mining site is located in the confined Auob Aquifer.
- ii. A 5-spot pattern is applied.
- iii. The total recharge to injection wells equals the discharge from the pumping well.
- iv. Natural groundwater flows towards an adjacent abstraction borehole located downstream of the mining site.

These assumptions help derive mathematical equations and build a conceptual mathematical model to describe groundwater flow and ISL operations under these conditions.

2.2.2 Why Such an Approach?

Mathematical models are classical tools employed to simulate and infer the action of real-world systems in scenarios where little or no empirical data is available. Since there are no long-term field data available from pilot tests of the Auob Aquifer, theoretical data were used in this study. Although theoretical models will never replace field validation, they give valuable information, particularly when the actual hydrogeological environment closely reflects the idealized conditions presented in the model (Fernandes et al. 2023).

2.2.3 Hydrogeological Context

The study area includes three aquifers (Table 1) namely:

1. Top: Kalahari Aquifer (unconfined aquifer)
2. Auob Aquifer: A confined aquifer in between, where ISL operations are proposed.
3. Nossob Aquifer: A bottom confined aquifer.

According to ORASECOM (2016), the thicknesses of the STAS units vary across the Basin, and they exhibit different hydraulic behaviours, as indicated by the hydraulic parameters below.

The Kalahari aquifer has an average thickness of about 150 m. The average borehole pumping rate is between 1.8 to 7.5 m³/h. Transmissivity is 30 m²/day.

The third impermeable layer is 30 m thick. But it is eroded at some parts causing the middle aquifer to behave like an unconfined aquifer.

The Middle aquifer is called the Auob aquifer where the mining will take place. It has a thickness of approximately 150 m. That means it is at a depth of 350 m. Transmissivity is ranging between 0.1 to 200 m²/day.

Table 1 Correlation of the hydrogeological and geological units of the Stampriet Transboundary Aquifer System (STAS) after Miller (2002)

Geological cross-section			Hydrogeological cross-section	
Kalahari Beds			Unconfined Kalahari Aquifer	
Karoo Supergroup	Rietmond Member	Upper	3rd Impermeable Layer	
		Lower	Auob Aquifer	
	Auob Member	A5		
		A4		
		A3		
		A2		
		A1		
	Mukorob Member	Upper	2nd Impermeable Layer	
		Lower	Nossob Aquifer	
	Nossob Member		1st Impermeable Layer	
Pre Eccia Group (Basement rocks of the Dwyka Group, the Nama Group and Damara Sequence)				

The borehole pumping rate (sustainable yield) is between 9.5 to 13.6 m³/h.

The 2nd impermeable layer is approximately 40 m thick.

The bottom aquifer is called the Nossob aquifer, and it has a thickness of 60 m and it is about 400 m deep. Transmissivity ranges from 0.02 to 7.01 m²/day.

The first impermeable layer is about 100 m thick.

Thus, the Auob Aquifer is an ideal scenario for specific modelling of ISL mining dynamics and water resource impacts. Due to the confined nature of the aquifer, it is crucial to also understand the effect of the cone of depression on adjacent boreholes.

2.2.4 Theoretical Modelling Framework

The subsequent simulations in this study represent a complex aquifer system, which includes a central abstraction borehole, multiple injection boreholes, and corresponding farmer boreholes. We understand that this is only a representation of a five-spot pattern of ISL mining, not the entire well-field of the mine. The actual number of injection and abstraction wells installed during the ISL mining in the Auob aquifer remains unknown. We will assume the presence of geological faults, which comprise the STAS, in the study area. We further assume a flat topography, simulating

a surface layer with ups and downs (light green areas) and a recharge zone, represented by arrows illustrating the path flow in Fig. 1. Generally, Fig. 1 represents how the layers and boreholes in a system interact with each other, making it easier to analyse the spatial structures and relationships among other elements. In a model of the split aquifer system, two rock layers serve as impermeable barriers above and below the first and second aquifers, respectively. We visualize the boreholes, which penetrate both the first and second aquifers, as vertical cylinders with their tops 2 cm above the surface level, ensuring their visibility and full coverage of both aquifers. The setup is specifically designed to investigate groundwater flow, well interference, and the management of aquifer resources. We assume groundwater flow along the x direction.

Moreover, Fig. 2 below shows a vertical cross-section through the system, highlighting the locations of the boreholes and their penetration into both the Kalahari and Auob aquifers. From the cross-section, we can show (clearly) where the surface, aquifer layers, and boreholes are located in relative positions. This view also shows how the fault zone influences the aquifer systems. The model changes the way we think about the retentiveness of solute transport on different scales, which makes it a better way to understand the hydrogeology of STAS. It simulates better large-scale systems like closely spaced boreholes, aquifers, and fault

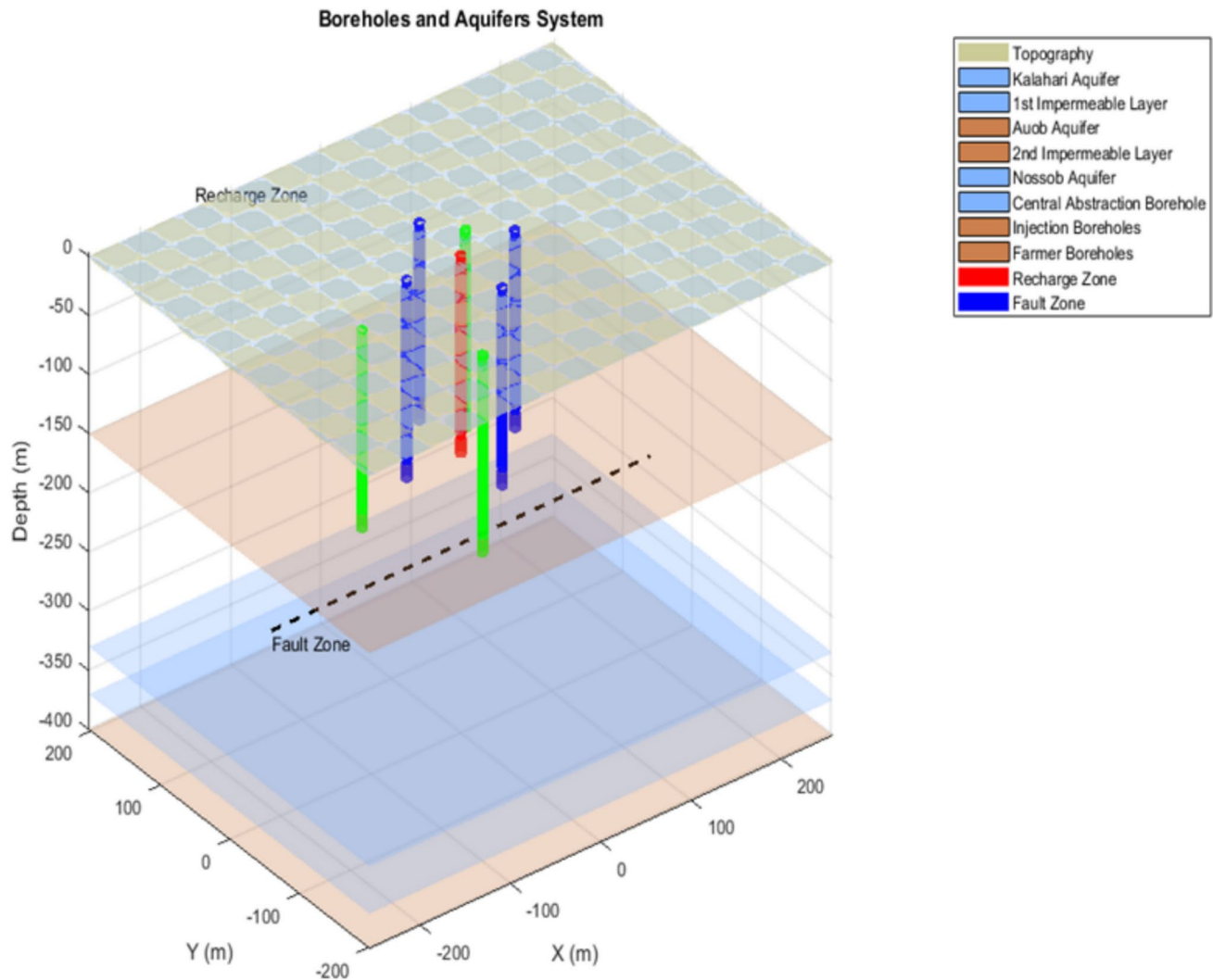


Fig. 1 A setup system of ISL abstraction (red), ISL injection (blue) and farmer's boreholes (green) in 3D view

zones to address applications such as interference studies, recharge zones, or variations of permeability due to faulting. Groundwater management, resource optimization, and environmental impact studies can utilize it, particularly in heterogeneous aquifer systems or active fault regions. The visual and interactive characteristics of the model serve as an educational and decision-support tool for hydrogeologists, environmental engineers, and water resource managers.

Therefore the model will be represented mathematically by considering:

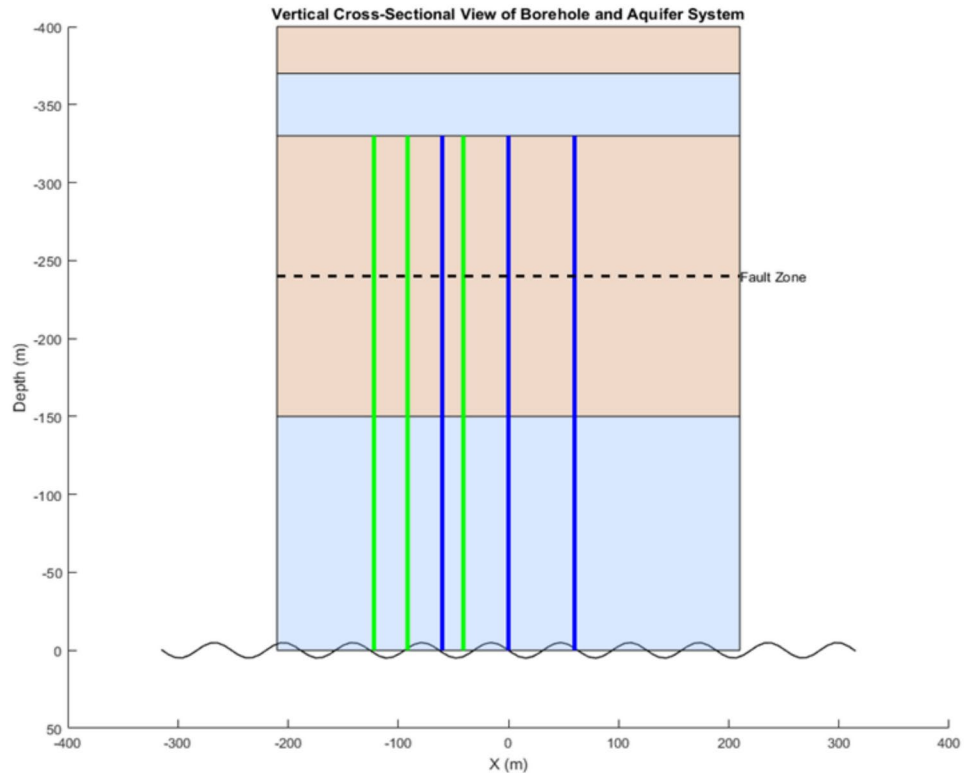
- The underlying 'drawdown effect' of excessive pumping from the Auob Aquifer.
- The interaction of the recharge from injection wells with the cone of depression.
- Dotted with deep buried abstraction boreholes for drawing the groundwater.

- This model will also include important parameters, such as hydraulic conductivity, storativity, and recharge rates, which will vary to accommodate uncertainties in site-specific conditions.

2.2.5 Relevance and Expected Results

This theoretical investigation not only points out the challenges and risks of ISL operations but also forms a solid framework for pilot tests in the future. Filling this knowledge gap, the study seeks to equip decision-makers with a more informed understanding of the hydrogeological consequences of ISL mining and provides insights for sustainable management practices of the Auob Aquifer and the STAS at large.

Fig. 2 Cross-section view of the setup system inclusive of the four injection wells and one abstraction wells and three production boreholes of the farmers



3 Mathematical Models for Groundwater Abstraction and Injection in a Confined Aquifer

As it has been mentioned earlier, this study focuses on modelling the interactions of the ISL well-fields (injection and abstraction wells) with the farmers' boreholes in the surrounding of the mining site. Thus, we shall consider a system of six boreholes, four injection boreholes surrounding an abstraction borehole, and a far abstraction borehole downstream of the 5-spot pattern well-field. All these boreholes are drilled and installed into the same confined aquifer. The far borehole belongs to a farmer, while the other boreholes form part of the well-field pattern used in in-situ leaching operations to extract uranium from the orebody in the same aquifer.

We denote the portion of the central injection boreholes (r_1, z_1) , (r_2, z_2) , (r_3, z_3) and (r_4, z_4) ; and the central borehole is (r_c^A, z_c^A) . While the farmer's borehole is denoted by (r_f, z_f) . Since we want to have the position of each borehole, we use the Delta-Dirac function. For injection boreholes, we shall have the contribution of each to be expressed by:

$$\frac{Q_{inj}}{T} \delta(r - r_1) \delta(z - z_1) \quad (1)$$

Note that $\delta(r - r_1) \delta(z - z_1)$ places each borehole at its respective location $(r_i - z_i)$. For the central borehole, we have its contribution to be represented by:

$$\frac{Q_c^{ex}}{T} \delta(r - r_c) \delta(z - z_c) \quad (2)$$

And for the farmer's borehole which is placed at (r_f, z_f) , its contribution will be:

$$\frac{Q_f^{ex}}{T} \delta(r - r_f) \delta(z - z_f) \quad (3)$$

If then $h(x, z, t)$ is the hydraulic head at the position (r, z) and time t , we will have the following equation:

$$\begin{aligned} & \frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial z^2} \\ & + \sum_{k=1}^4 \frac{Q_c^k}{T} \delta(r_i - r_k) \delta(z_j - z_k) \\ & - \frac{Q_c^{ex}}{T} \delta(r_i - r_c) \delta(z_j - z_c) \\ & - \frac{Q_f^{ex}}{T} \delta(r_i - r_f) \delta(z_j - z_f) \\ & = S \frac{\partial h}{\partial t} \end{aligned} \quad (4)$$

The above equation can be generalized where the system has many far extraction boreholes representing farmers' boreholes and also a replica of many 5-spot patterns forming a well field.

The above Eq. (4) can be solved using the Crank-Nicolson numerical scheme, whereby;

$$\begin{aligned}\frac{\partial h}{\partial t} &\simeq \frac{h_{i,j}^{n+1} - h_{i,j}^n}{\Delta t}, \quad h_{i,j}^{n+1} \simeq h(r_i, z_j, t_{n+1}), \\ \frac{\partial^2 h}{\partial z^2} &\simeq \frac{(h_{i,j+1}^{n+1} - 2h_{i,j}^{n+1} + h_{i,j-1}^{n+1} + h_{i,j+1}^n - 2h_{i,j}^n + h_{i,j-1}^n)}{2\Delta z^2}, \\ \frac{\partial^2 h}{\partial r^2} &\simeq \frac{(h_{i+1,j}^{n+1} - 2h_{i,j}^{n+1} + h_{i-1,j}^{n+1} + h_{i+1,j}^n - 2h_{i,j}^n + h_{i-1,j}^n)}{2\Delta r^2}, \\ \frac{\partial h}{\partial r} &\simeq \frac{(h_{i+1,j}^{n+1} - h_{i-1,j}^{n+1} + h_{i+1,j}^n - h_{i-1,j}^n)}{2\Delta r}\end{aligned}$$

Replacing these terms in Eq. (4), we obtain the following scheme.

$$\begin{aligned}&\frac{1}{2\Delta r^2} \{h_{i,j+1}^{n+1} - 2h_{i,j}^{n+1} + h_{i,j-1}^{n+1} + h_{i,j+1}^n - 2h_{i,j}^n + h_{i,j-1}^n\} \\ &+ \frac{1}{2r_i\Delta r} \{h_{i+1,j}^{n+1} - h_{i-1,j}^{n+1} + h_{i+1,j}^n - h_{i-1,j}^n\} \\ &+ \frac{1}{2\Delta z^2} \{h_{i+1,j}^{n+1} - 2h_{i,j}^{n+1} + h_{i-1,j}^{n+1} + h_{i+1,j}^n - 2h_{i,j}^n + h_{i-1,j}^n\} \\ &+ \sum_{k=1}^4 \frac{Q_c^k}{T} \delta(r_i - r_k) \delta(z_j - z_k) - \frac{Q_c^{ex}}{T} \delta(r_i - r_c) \delta(z_j - z_c) \\ &- \frac{Q_f^{ex}}{T} \delta(r_i - r_f) \delta(z_j - z_f) \\ &= S \frac{h_{i,j}^{n+1} - h_{i,j}^n}{\Delta t}\end{aligned}\quad (5)$$

3.1 Stability Analysis Via von Neumann

We now assume that the error is given by $E_{i,j}^n$ at each grid point that represents the Fourier node as:

$$E_{i,j}^n = \delta_n \exp(i[k_r r_i + k_z z_j]) \quad (6)$$

Substituting the Fourier node into the equation yields;

(a) For the time term

$$\frac{S}{\Delta t} (\delta_{n+1} \exp(i[k_r r_i + k_z z_j]) - \delta_n \exp(i[k_r r_i + k_z z_j])) \quad (7)$$

By simplifying, we get;

$$\frac{S}{\Delta t} (\delta_{n+1} - \delta_n) \exp(i[k_r r_i + k_z z_j]) \quad (8)$$

(b) And for the Crank-Nicolson radial term, we have the following for the second derivative;

$$\frac{h_{i+1,j}^{n+1} - 2h_{i,j}^{n+1} + h_{i-1,j}^{n+1}}{2\Delta r^2} - \frac{h_{i+1,j}^n - 2h_{i,j}^n + h_{i-1,j}^n}{2\Delta r^2} = \prod_{i,j}^n \quad (9)$$

Will become,

$$\begin{aligned}&\delta_{n+1} \exp(i[k_r (r_i + \Delta r) + k_z z_j]) \\ &- 2\delta_{n+1} \exp(i[k_r r_i + k_z z_j]) \\ &+ \delta_{n+1} \exp(i[k_r (r_i - \Delta r) + k_z z_j]) \\ &= \delta_{n+1} \exp(i[k_r (r_i - \Delta r) + k_z z_j]) \\ &(\exp(i[k_r \Delta r]) + \exp(-i[k_r \Delta r]) - 2)\end{aligned}\quad (10)$$

Then,

$$\begin{aligned}&\prod_{i,j}^n \\ &= \frac{1}{2(\Delta r)^2} (\delta_{n+1} + \delta_n) \exp(i[k_r r_i + k_z z_j]) \\ &(\exp(i[k_r \Delta r]) + \exp(-i[k_r \Delta r]) - 2) \\ &= \frac{\exp(i[k_r r_i + k_z z_j])}{(\Delta r)^2} (\delta_{n+1} + \delta_n) (\cos(\Delta r) - 1)\end{aligned}\quad (11)$$

For the velocity term, we have,

$$\begin{aligned}&\frac{1}{2r_i} \left(\frac{h_{i+1,j}^{n+1} - h_{i-1,j}^{n+1}}{2\Delta r} + \frac{h_{i+1,j}^n - h_{i-1,j}^n}{2\Delta r} \right) \\ &= \frac{\exp(i[k_r r_i + k_z z_j]) \operatorname{isin}(k_r \Delta r) (\delta_{n+1} + \delta_n)}{\Delta r r_i}\end{aligned}\quad (12)$$

(c) For the vertical term,

$$\frac{\partial^2 h}{\partial z^2} \simeq -\frac{1}{(\Delta z)^2} (\delta_{n+1} + \delta_n) (\cos(k_z \Delta z) - 1) \exp(i[k_r r_i + k_z \Delta z]) \quad (13)$$

Therefore, the full equation yields,

$$\begin{aligned}S \frac{\delta_{n+1} - \delta_n}{\Delta t} &= \left\{ -T_r \frac{1 - \cos(k_r \Delta r)}{(\Delta r)^2} (\delta_{n+1} + \delta_n) \right. \\ &+ T_r \frac{\operatorname{isin}(k_r \Delta r)}{r_i \Delta r} (\delta_{n+1} + \delta_n) \\ &\left. - T_z \frac{1 - \cos(k_z \Delta z)}{(\Delta z)^2} (\delta_{n+1} + \delta_n) \right\}\end{aligned}\quad (14)$$

By introducing the amplification factor A , we note that for the condition of stability $|A| < 1$. Thus from the above Eq. (14), we get the following:

$$\frac{S}{\Delta t} (A - 1) = A \left\{ -T_r \frac{1 - \cos(k_r \Delta r)}{(\Delta r)^2} (A + 1) + T_{ri} \frac{\text{isin}(k_r \Delta r)}{r_i \Delta r} (A + 1) - T_z \frac{1 - \cos(k_z \Delta z)}{(\Delta z)^2} (A + 1) \right\} \quad (15)$$

So for simplification, we let,

$$\Omega_r = \frac{T_r}{(\Delta r)^2} (1 - \cos(k_r \Delta r))$$

$$\Omega_r^1 = \frac{T_r}{r_i \Delta r} (\text{isin}(k_r \Delta r))$$

$$\Omega_z = \frac{T_r}{(\Delta z)^2} (1 - \cos(k_z \Delta z))$$

Replacing these terms into Eq. (15), we obtain;

$$\frac{S}{\Delta t} (A - 1) = A (A + 1) (-\Omega_r + \Omega_r^1 - \Omega_z) \quad (16)$$

We can therefore reformulate Eq. (16) to produce,

$$\left(\frac{S}{\Delta t} + \Omega_r - \Omega_r^1 + \Omega_z \right) A - \frac{S}{\Delta t} + A^2 (\Omega_r - \Omega_r^1 + \Omega_z) = 0 \quad (17)$$

If we let $b = \Omega_r - \Omega_r^1 + \Omega_z + \frac{S}{\Delta t}$, $C = \frac{S}{\Delta t}$ and $a = \Omega_r - \Omega_r^1 + \Omega_z$; then Eq. (17) can be simplified as:

$$aA^2 - bA + C = 0 \quad (18)$$

Noting that the above Eq. (18) contains real and imaginary parts. Therefore; $aA^2 - bA + C = 0$ if the real and imaginary parts are then zero. Therefore for the imaginary part, we have;

$$-A^2 \Omega_r^1 - A \Omega_r^1 = 0 \quad \text{if } A = 1 \text{ or } A = 0 \quad (19)$$

But we need $|A| < 1$, therefore this term a is not considered.

For the real part, we have;

$$A^2 (\Omega_r + \Omega_z) - A \left(\frac{S}{\Delta t} + \Omega_r + \Omega_z \right) + \frac{S}{\Delta t} = 0 \quad (20)$$

Therefore,

$$A = \frac{\left(\frac{S}{\Delta t} + \Omega_r + \Omega_z \right) \pm \sqrt{\left(\frac{S}{\Delta t} + \Omega_r + \Omega_z \right)^2 - 4 (\Omega_r + \Omega_z) \frac{S}{\Delta t}}}{2 (\Omega_r + \Omega_z)} \quad (21)$$

We note that:

$$\begin{aligned} & \left(\frac{S}{\Delta t} + \Omega_r + \Omega_z \right)^2 - 4 (\Omega_r + \Omega_z) \frac{S}{\Delta t} \\ &= \left(\frac{S}{\Delta t} \right)^2 - 2 \frac{S}{\Delta t} (\Omega_r + \Omega_z) + (\Omega_r + \Omega_z)^2 \end{aligned}$$

Thus, A becomes;

$$A = \frac{\left(\frac{S}{\Delta t} + \Omega_r + \Omega_z \right) \pm \sqrt{\left(\frac{S}{\Delta t} + \Omega_r + \Omega_z \right)^2 - 2 (\Omega_r + \Omega_z) \frac{S}{\Delta t} + (\Omega_r + \Omega_z)^2}}{2 (\Omega_r + \Omega_z)} \quad (22)$$

We would like to have a condition for which $|A| < 1$. However, from the formula of A , we have that;

$$A = \frac{\left(\frac{S}{\Delta t} + \Omega_r + \Omega_z \right) \pm \left(\frac{S}{\Delta t} - \Omega_r - \Omega_z \right)}{2 (\Omega_r + \Omega_z)} \quad (23)$$

So,

$$A_+ = \frac{2 \frac{S}{\Delta t}}{2 (\Omega_r + \Omega_z)}, \quad A_- = \frac{\frac{S}{\Delta t} + 2 (\Omega_r + \Omega_z)}{2 (\Omega_r + \Omega_z)} \quad \text{such that } |A| < 1$$

if $\frac{S}{\Delta t (\Omega_r + \Omega_z)} < 1$. Therefore, we will need $\frac{S}{(\Omega_r + \Omega_z)} < \Delta t$.

Now the condition is that;

$$\frac{S}{T_r \frac{1 - \cos(k_r \Delta r)}{(\Delta r)^2} + T_z \frac{1 - \cos(k_z \Delta z)}{(\Delta z)^2}} < \Delta t \quad (24)$$

3.2 Truncation Error

In this section, the truncation error is obtained using the Taylor series approximation.

(1) Time term

$$h_{i,j}^{n+1} = h_{i,j}^n + \Delta t \frac{\partial h}{\partial t} \Big|_{t^n} + \frac{\Delta t^2}{2} \frac{\partial^2 h}{\partial t^2} + O(\Delta t^3) \quad (25)$$

The forward Euler approximation for $\frac{\partial h}{\partial t}$ is $h_{i,j}^{n+1} - h_{i,j}^n \simeq \Delta t \frac{\partial h}{\partial t} \Big|_{t=t^n}$, which leads to the truncation error to be $O(\Delta t^2)$.

(2) For the first derivative radial term

$$h_{i+1,j}^n = h_{i,j}^n + \Delta r \frac{\partial h}{\partial r} \Big|_{r=r_i} + \frac{\Delta r^2}{2} \frac{\partial^2 h}{\partial r^2} \Big|_{r=r_i} + O(\Delta r^3) \quad (26)$$

$$h_{i-1,j}^n = h_{i,j}^n - \Delta r \frac{\partial h}{\partial r} \Big|_{r=r_i} + \frac{\Delta r^2}{2} \frac{\partial^2 h}{\partial r^2} \Big|_{r=r_i} + O(\Delta r^3) \quad (27)$$

$$h_{i+1,j}^{n+1} = h_{i,j}^{n+1} + \Delta r \frac{\partial h}{\partial r} \Big|_{r=r_i} + \frac{\Delta r^2}{2} \frac{\partial^2 h}{\partial r^2} \Big|_{r=r_i} + O(\Delta r^3) \quad (28)$$

$$h_{i-1,j}^{n+1} = h_{i,j}^{n+1} - \Delta r \left. \frac{\partial h}{\partial r} \right|_{r=r_i} + \frac{\Delta r^2}{2} \left. \frac{\partial^2 h}{\partial r^2} \right|_{r=r_i} + O(\Delta r^3) \quad (29)$$

Replacing these with the central difference leads to:

$$\begin{aligned} \left. \frac{\partial h}{\partial r} \right|_{r=r_i} &\simeq \frac{1}{4\Delta r} \left\{ \left[h_{i,j}^{n+1} + \Delta r \left. \frac{\partial h}{\partial r} \right|_{r=r_i} + O(\Delta r^2) \right] \right. \\ &\quad - \left[h_{i,j}^{n+1} - \Delta r \left. \frac{\partial h}{\partial r} \right|_{r=r_i} + O(\Delta r^2) \right] \\ &\quad + \left[h_{i,j}^n + \Delta r \left. \frac{\partial h}{\partial r} \right|_{r=r_i} + O(\Delta r^2) \right] \\ &\quad - \left. \left[h_{i,j}^n - \Delta r \left. \frac{\partial h}{\partial r} \right|_{r=r_i} + O(\Delta r^2) \right] \right\} \\ &= \frac{1}{2\Delta r} \left(2\Delta r \left. \frac{\partial h}{\partial r} \right|_{r=r_i} \right) + O(\Delta r^2) \\ &= \left. \frac{\partial h}{\partial r} \right|_{r=r_i} + O(\Delta r^2) \end{aligned} \quad (30)$$

(3) And for the second derivative radial term, we shall have;

$$\begin{aligned} \left. \frac{\partial^2 h}{\partial r^2} \right|_{r=r_i} &\simeq \frac{1}{2\Delta r^2} \left\{ \left[h_{i,j}^{n+1} + \Delta r \left. \frac{\partial h}{\partial r} \right|_{r=r_i} + \frac{\Delta r^2}{2} \left. \frac{\partial^2 h}{\partial r^2} \right|_{r=r_i} \right] \right. \\ &\quad - 2h_{i,j}^{n+1} + \left[h_{i,j}^{n+1} - \Delta r \left. \frac{\partial h}{\partial r} \right|_{r=r_i} + \frac{\Delta r^2}{2} \left. \frac{\partial^2 h}{\partial r^2} \right|_{r=r_i} \right] \\ &\quad + \left[h_{i,j}^n + \Delta r \left. \frac{\partial h}{\partial r} \right|_{r=r_i} + \frac{\Delta r^2}{2} \left. \frac{\partial^2 h}{\partial r^2} \right|_{r=r_i} \right] \\ &\quad - 2h_{i,j}^n + \left. \left[h_{i,j}^n - \Delta r \left. \frac{\partial h}{\partial r} \right|_{r=r_i} + \frac{\Delta r^2}{2} \left. \frac{\partial^2 h}{\partial r^2} \right|_{r=r_i} \right] \right\} \\ &= \frac{1}{2\Delta r^2} (2h_{i,j}^{n+1} - 2h_{i,j}^{n+1} + 2h_{i,j}^{n+1} - 2h_{i,j}^n + 2O(\Delta r^2)) \\ &= \left. \frac{\partial^2 h}{\partial r^2} \right|_{r=r_i} + O(\Delta r^2) \end{aligned} \quad (31)$$

(4) In the same way for the vertical term, we get;

$$\left. \frac{\partial^2 h}{\partial z^2} \right|_{r=r_i} \simeq \left. \frac{\partial^2 h}{\partial z^2} \right|_{z=z_j} + O(\Delta z^2) \quad (32)$$

Therefore, the total truncation error is given by:

$$T = O(\Delta r^2 + \Delta z^2 + \Delta t^2) \quad (33)$$

Error scale as from the above total truncation error, one can conclude that the time derivative error accumulates as $O(\Delta t^2)$ per timestep. However, over t_n times steps, the $MO(\Delta t) = O(M\Delta t)$. This error could be significant for a longer simulation when Δt is larger. On the other hand for the radial and vertical terms, over time the error can accumulate in particular with sharp gradients near boreholes.

3.3 Contribution of Each Borehole

In this section, we shall present the contribution of each borehole.

(1) For an injection borehole situated at (r_i, z_j) , the hydraulic head contribution at the point (r, z) is given by:

$$h_i^{inj}(r, z, t) = \frac{Q_i^{inj}}{2\pi T} W \left(\frac{(r - r_i)^2 - (z - z_i)^2}{4TtS} \right) \quad (34)$$

where W is the well-function defined by

$$W(\mu) = \int_{\mu}^{\infty} \frac{\exp(-l)}{l} dl$$

(2) For the central abstraction borehole, we have;

$$h_c^{ext}(r, z, t) = -\frac{Q_c^{ext}}{2\pi T} W \left(\frac{r^2 + z^2}{4TtS} \right) \quad (35)$$

The negative sign implies that this borehole reduces the water level.

(3) While for the far borehole which extracts water at point (r_f, z_f) , the hydraulic head contributes at the point (r, z) is also negative. Thus,

$$h_f^{ext}(r, z, t) = -\frac{Q_f^{ext}}{4\pi T} W \left(\frac{(r - r_f)^2 - (z - z_f)^2}{4TtS} \right) \quad (36)$$

Therefore, the superposition of hydraulic heads at any points (r, z) is given as:

$$h_T(r, z, t) = \sum_{j=1}^4 h_i^{inj}(r, z, t) - h_c^{ext}(r, z, t) - h_f^{ext}(r, z, t) \quad (37)$$

3.4 Sensitivity Analysis on S and T

In this section, we aim to investigate the sensibility of the total hydraulic head as a function of S and T . Since the total is a sum, we shall investigate it step-by-step.

If,

$$h_i^{inj}(S) = \frac{Q_i^{inj}}{4\pi T} W \left(\frac{(r - r_i)^2 + (z - z_j)^2}{4TtS} \right) \quad (38)$$

$$\text{We let, } \mu(S) = \frac{(r - r_i)^2 + (z - z_j)^2}{4TtS}$$

Therefore,

$$\frac{d\mu}{dS} = -\frac{(r-r_i)^2 + (z-z_j)^2}{4TtS^2} \quad (39)$$

And

$$\frac{dh_i^{inj}}{dS} = \frac{Q_i^{inj}}{4\pi T} \frac{d}{d\mu} W(\mu) \quad (40)$$

So Eq. (38) can be rewritten as;

$$\frac{d\mu}{dS} = -\frac{Q_i^{inj}}{4\pi T} \frac{d}{d\mu} W(\mu) \left(-\frac{(r-r_i)^2 + (z-z_j)^2}{4TtS^2} \right) \quad (41)$$

Then, if $\frac{d(W(\mu))}{d\mu} = -\frac{\exp(-\mu)}{\mu}$, we substitute it into Eq. (41) and simplify. Therefore,

$$\begin{aligned} \frac{dh_i^{inj}}{dS} &= \frac{Q_i^{inj}}{4\pi T} \left(\frac{\exp(-\mu)}{\frac{(r-r_i)^2 + (z-z_j)^2}{4TtS^2}} \right) \left(-\frac{(r-r_i)^2 + (z-z_j)^2}{4TtS^2} \right) \\ &= -\frac{Q_i^{inj}}{4\pi ST} \exp(-\mu) \end{aligned} \quad (42)$$

$$\frac{dh_c^{ext}}{dS} = -\frac{Q_c^{ext}}{4\pi ST} \exp(-\mu) \quad (43)$$

$$\frac{dh_f^{ext}}{dS} = -\frac{Q_f^{ext}}{4\pi ST} \exp(-\mu) \quad (44)$$

Therefore,

$$\frac{dh(S)}{dS} = -\frac{\exp(-\mu)}{4\pi T} [Q_i^{inj} + Q_c^{ext} + Q_f^{ext}] \quad (45)$$

We can see that $\frac{dh(S)}{dS} < 0$.

We further note that;

$$\frac{d^2 h_r}{dS^2} = \frac{d^2 h_c^{ext}}{dS^2} + \frac{d^2 h_f^{ext}}{dS^2} + \frac{d^2}{dS^2} \left(\sum_{i=1}^4 h_i^{inj} \right) \quad (46)$$

$$\frac{d^2}{dS^2} h_i^{inj}(S) = \frac{Q_i^{inj}}{4\pi T} \left(\frac{d^2 W(\mu)}{d\mu^2} \left(\frac{d\mu}{dS} \right)^2 + \frac{dW(\mu)}{d\mu} \frac{d^2 \mu}{dS^2} \right) \quad (47)$$

$$\frac{d^2 \mu}{dS^2} = \frac{2(r-r_i)^2 + (z-z_j)^2}{4TtS^3} \quad (48)$$

The sign of the second derivative is positive since is the sum of positive terms. This is the same with transmissivity, since for example;

$$\frac{d}{dT} h_i^{ext} = -\frac{Q_c^{ext}}{4\pi T^2} W(\mu) - Q_c^{ext} \frac{dW(\mu)}{d\mu} \left(\frac{-(r-r_i)^2 + (z-z_j)^2}{4T^2 t S} \right) \quad (49)$$

And the second derivative;

$$\begin{aligned} \frac{d^2}{dT^2} h_i^{ext} &= -\frac{Q_c^{ext}}{4\pi} \left(\frac{2W(\mu)}{T^3} + \frac{d^2 W(\mu)}{d\mu^2} \cdot \left(\frac{d(\mu)}{dT} \right)^2 \right) \\ &\quad + \frac{2}{T^2} \frac{dW(\mu)}{d\mu} \cdot \frac{d(\mu)}{dT} \end{aligned} \quad (50)$$

Thus,

$$\frac{dh_{total}}{dT} = \sum_{i=1}^4 \frac{dh_i^{inj}}{dT} - \frac{dh_c^{ext}}{dT} - \frac{dh_f^{ext}}{dT} < 0 \quad (51)$$

$$\frac{d^2 h_{total}}{dT^2} = \sum_{i=1}^4 \frac{d^2 h_i^{inj}}{dT^2} - \frac{d^2 h_c^{ext}}{dT^2} - \frac{d^2 h_f^{ext}}{dT^2} > 0 \quad (52)$$

3.5 Asymptotic Behaviour: Early and Late Times

In order to understand the transient and steady-state behaviour of the groundwater flow, we shall analyse the asymptotic behaviour of the total hydraulic head $h_{total}(r, z, t)$ for early and late times.

(i) For early time, that is when $t \rightarrow 0$, we recall that the well-function will behave as follows;

$$W\left(\frac{r^2 + z^2}{4TtS}\right) \simeq \frac{r^2 + z^2}{8TtS} \quad (53)$$

This is due to the fact that $W(\mu) \simeq \frac{\mu}{2}$ for the condition that $\mu \ll 0.5$.

Now for the injection boreholes i , we have:

$$\begin{aligned} h_i^{inj}(r, z, t) &\simeq \frac{Q_i^{inj}}{4\pi T} \left(\frac{(r-r_i)^2 + (z-z_j)^2}{8TtS} \right) \\ &= \frac{Q_i^{inj}}{32\pi T^2 t S} \left\{ (r-r_i)^2 + (z-z_j)^2 \right\} \end{aligned} \quad (54)$$

One will notice that the head contribution decreases at t^{-1} and is proportional to the distance squared distances from the observation point to the place where the borehole is placed.

For the central borehole, we have;

$$h_c^{ext}(r, z, t) \simeq -\frac{Q_c^{ext}}{32\pi T^2 t S} \left\{ (r - r_i)^2 + (z - z_i)^2 \right\} \quad (55)$$

The total hydraulic head for early-time behaviour is then;

$$h_{total}(r, z, t) \simeq -\frac{1}{32\pi T^2 t S} \left\{ \sum_{j=1}^4 Q_i^{inj} \left((r - r_i)^2 + (z - z_i)^2 \right) - Q_c^{ext} \left((r - r_c)^2 + (z - z_c)^2 \right) - Q_f^{ext} \left((r - r_f)^2 + (z - z_f)^2 \right) \right\} \quad (56)$$

Therefore, $h_{total}(r, z, t)$ is proportional to $\frac{1}{t}$ as $t \rightarrow 0$. This is in agreement with expectations from the well-known transient flow theory.

(ii) For late time, it is well-known that $W(\mu) \simeq -\ln \mu - 0.5772$ as $\mu \rightarrow \infty$. Therefore,

$$W \left(\frac{(r - r_i)^2 + (z - z_i)^2}{4TtS} \right) \simeq -\ln \left\{ \frac{(r - r_i)^2 + (z - z_i)^2}{4TtS} \right\} - 0.5772 \quad (57)$$

Adding the contribution of each borehole we obtain;

$$h_{total}(r, z, t) \simeq -\frac{1}{4\pi T} \left\{ \sum_{j=1}^4 Q_i^{inj} \ln \left(\frac{4TtS}{(r - r_i)^2 + (z - z_i)^2} \right) - Q_c^{ext} \ln \left(\frac{4TtS}{(r - r_c)^2 + (z - z_c)^2} \right) - Q_f^{ext} \ln \left(\frac{4TtS}{(r - r_f)^2 + (z - z_f)^2} \right) \right\} \quad (58)$$

The conclusion is that for late time, the hydraulic head $h_{total}(r, z, t)$ decays logarithmically with time as $t \rightarrow \infty$.

3.6 Analysis of Radius of Influence

We aim at investigating the radius of influence between the far borehole and the central borehole taking into account the effect of injection boreholes. To do this, we will use the solution at the later time. We start with the injection boreholes. Thus for each injection borehole i the distance from the central borehole is $(r_i - r_c)$, so;

$$h_i^{inj}(r_c, z, t) \simeq -\frac{Q_i^{inj}}{4\pi T} \ln \left(\frac{(r_c - r_i)^2 + (z - z_j)^2}{4Tt} \right) \quad (59)$$

The sum of the contribution from the four injection boreholes is given as:

$$h_{total}^{inj}(r_c, z, t) = \sum_{k=1}^4 h_k^{inj}(r_c, z, t) \simeq \frac{1}{4\pi T} \sum_{k=1}^4 Q_k^{inj} \left[-\ln \left(\frac{(r_c - r_k)^2 + (z - z_k)^2}{4TtS} \right) \right] \quad (60)$$

We should combine the effect of both central and injection boreholes. Such that;

$$h_{total}^{central}(r_c, z, t) = h_c^{ext}(r_c, z, t) + \sum_{i=1}^4 h_i^{inj}(r_c, z, t) = -\frac{Q_c^{ext}}{4\pi T} \ln \left(\frac{(r_c)^2 + (z_c)^2}{4TtS} \right) + \frac{1}{4\pi T} \sum_{i=1}^4 Q_i^{inj} \left[-\ln \left(\frac{(r_c - r_i)^2 + (z_c - z_k)^2}{4TtS} \right) \right] = \frac{1}{4\pi T} \left[-Q_c^{ext} \ln \left(\frac{r_c^2 + z_c^2}{4TtS} \right) + \sum_{i=1}^4 Q_i^{inj} \left[-\ln \left(\frac{(r_c - r_i)^2 + (z_c - z_k)^2}{4TtS} \right) \right] \right] \quad (61)$$

Let $R_{ext,c}^{inj}$ be the radius of influence of the central borehole. Thus radius is the radial distance from the central borehole where the hydraulic head will decay to a threshold value $h^{threshold}$. In mathematical terms, this is:

$$h^{central}(R_{ext,c}^{inj}, z, t) = h^{threshold} \quad (62)$$

So,

$$\frac{1}{4\pi r} \left[-Q_c^{ext} \ln \left(\frac{(R_{ext,c}^{inj})^2 + (z)^2}{4TtS} \right) + \sum_{i=1}^4 Q_i^{inj} \left[-\ln \left(\frac{(R_{ext,c}^{inj} - r_i)^2 + (z - z_i)^2}{4TtS} \right) \right] \right] = h^{threshold} \quad (63)$$

After manipulation, we obtain;

$$(R_{ext,c}^{inj})^2 + z^2 = 4TtS \exp \left(\frac{-4\pi r h^{threshold}}{Q_c^{ext}} \right) \prod_{i=1}^4 \left(\frac{(R_{ext,c}^{inj} - r_i)^2 + (z - z_i)^2}{4TtS} \right) \frac{Q_i^{inj}}{Q_c^{ext}} \quad (64)$$

$$\left(R_{ext,c}^{inj}\right) = \sqrt{4TtS.exp\left(\frac{-4\pi rh^{threshold}}{Q_c^{ext}}\right) \prod_{i=1}^4 \left(\left(\frac{\left(R_{ext,c}^{inj} - r_i\right)^2 + (z - z_i)^2}{4TtS}\right) \frac{Q_i^{inj}}{Q_c^{ext}}\right) - z^2} \quad (65)$$

The condition is that;

$$4TtS.exp\left(\frac{-4\pi rh^{threshold}}{Q_c^{ext}}\right) \prod_{i=1}^4 \left(\left(\frac{\left(R_{ext,c}^{inj} - r_i\right)^2 + (z - z_i)^2}{4TtS}\right) \frac{Q_i^{inj}}{Q_c^{ext}}\right) > z^2$$

That is to say;

$$\prod_{i=1}^4 \left(\left(\frac{\left(R_{ext,c}^{inj} - r_i\right)^2 + (z - z_i)^2}{4TtS}\right) \frac{Q_i^{inj}}{Q_c^{ext}}\right) > \frac{z^2.exp\left(\frac{-4\pi rh^{threshold}}{Q_c^{ext}}\right)}{4TtS}$$

To calculate $R_{ext,c}^{inj}$, we use the Newton–Raphson formula; thus:

$$R_{n+1} = R_n - \frac{f(R_n)}{f'(R_n)} \quad (66)$$

where, $R_n = R_{ext,c}^{inj}$.

3.7 Mass Balance and Flow Analysis of the Model

We will provide a flow and mass balance analysis that helps to determine the net flow. Here we are still under the condition that the injection borehole uses water extracted from the central borehole. However, due to evaporation from the evaporation ponds and the harsh climatic conditions of the study area, we assume there might be a 10% less reduction. So we let $\alpha \in [0,1]$, when α is the efficiency of the transfer system. If then $\alpha = 1$, then the designed system is perfectly efficient. But if $\alpha < 1$ then there is a proportion of extracted water by the central borehole that is lost and only αQ_c^{ext} would be available for re-injection. Therefore, the total losses are given as $(1 - \alpha) Q_c^{ext}$.

If Q_{total}^{inj} is the total injection volume from four boreholes, then $Q_{total}^{inj} = \sum_{i=1}^4 Q_i^{inj} = \alpha |Q_c^{ext}|$. Therefore each borehole receives;

$$Q_i^{inj} = \frac{\alpha}{4} |Q_c^{ext}| \quad (67)$$

The total net flow with losses is then calculated as;

$$\begin{aligned} Q^{net} &= Q_{total}^{inj} - (|Q_c^{ext}| + |Q_f^{ext}|) \\ &= \alpha |Q_c^{ext}| - (|Q_c^{ext}| + |Q_f^{ext}|) \\ &= (\alpha - 1) |Q_c^{ext}| - |Q_f^{ext}| \end{aligned} \quad (68)$$

If $\alpha = 1$, then,

$$Q^{net} = -|Q_f^{ext}| \quad (69)$$

And if we assume that, the transfer system is not running properly, thus $\alpha = 0$, then,

$$Q^{net} = -|Q_c^{ext}| - |Q_f^{ext}| \quad (70)$$

3.8 Re-injection Effect with Transfer Efficiency

In this section, we shall include in the mathematical model the effect of re-injection with α . Since,

$$Q_i^{inj} = \frac{\alpha}{4} |Q_c^{ext}| \quad (71)$$

Then the modified groundwater flow in radial coordinates becomes;

$$\begin{aligned} &\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial h}{\partial r} \right) \\ &+ \frac{\partial^2 h}{\partial z^2} + \sum_{i=1}^4 \frac{\alpha |Q_c^{ext}|}{4T} \delta(r - r_i) \delta(z - z_i) \\ &- \frac{|Q_c^{ext}|}{T} \delta(r - r_c) \delta(z - z_c) \\ &- \frac{|Q_f^{ext}|}{T} \delta(r - r_f) \delta(z - z_f) \\ &= S \frac{\partial h}{\partial t} \end{aligned} \quad (72)$$

The superposition hydraulic head is given by;

$$h_{total}(r, z, t) = \sum_{i=1}^4 h_i^{inj}(r, z, t) - h_i^{ext}(r, z, t) - h_f^{ext}(r, z, t) \quad (73)$$

$$\text{where, } h_i^{inj}(r, z, t) = \frac{\alpha |Q_c^{ext}|}{4\pi T} W\left(\frac{(r - r_i)^2 + (z - z_i)^2}{4TtS}\right)$$

Now if we consider the function $\Gamma(r, z, \bar{r}, \bar{z}, \bar{t})$ to be the sheen function that stand for the impact of a point source placed at $(\bar{r}, \bar{z})$ on the hydraulic head at (r, z) . Then,

$$\begin{aligned}
 h(r, z, t) = & h_0 - \frac{|Q_c^{ext}|}{4\pi T} \Gamma(r, z, \bar{r}, \bar{z}, \bar{t}) \\
 & + \frac{\alpha |Q_c^{ext}|}{4} \sum_{i=1}^4 \Gamma(r, z, r_i, z_i, t) \\
 & - \frac{|Q_c^{ext}|}{4\pi T} \Gamma(r, z, r_f, z_f, t)
 \end{aligned} \quad (74)$$

where;

$$\Gamma(r, z, \bar{r}, \bar{z}, \bar{t}) = \int_0^{\bar{t}} \frac{1}{4\pi l} \exp\left(\frac{-(r - \bar{r})^2 + (z - \bar{z})^2}{4Tl/S}\right) dl$$

3.9 Numerical Simulations and Results

The abstraction and total injection rates for the in-situ leach (ISL) mining for uranium in the Auob aquifer from the pumping well in a five-spot pattern were set to be 386 m³/day and 378 m³/day respectively. The three farmers' boreholes production yield was chosen to be 10 m³/day, 12 m³/day and 15 m³/day. The radius of influence or the distance between the ISL pumping well and the farmers' boreholes is 100 m. The static water level was set to be 12 m below surface level, assuming that the wells are not free-flowing. During simulations, we assumed that both the mine and the farmers would be operating at the same period for 12 h from 6 AM to 6 PM for one year. All wells were assumed to fully

penetrate the Auob aquifer at 330 m depth. Since groundwater recharge in Namibia is estimated to be one percent of the maximum annual precipitation (300 mm in the study area) and evapotranspiration is about 90%, then these parameters were allocated the approximate values of 0.00003 m/day and 0.0027 m/day respectively. The simulation results are shown in the Figures presented hereafter. Figures 3 and 4 show that evapotranspiration in the study area is intense across the hydrological year, indicating that very limited or zero recharge reaches the Auob aquifer.

It is observed that with time the total volume of water abstracted from the Auob aquifer within 12 months during the in-situ leach mining will be more than the total volume of water injected into the aquifer as shown in Fig. 5.

A significant loss is estimated to occur during the ISL operations due to increased evaporation from the evaporation ponds, although water may be injected into the system (Fig. 6). This means that the amount of water that will be re-injected during each cycle will reduce with time, regardless of its positive impact on the water table. Therefore, a large margin of losses could be incurred if the operation lasts longer under the prevailing climate conditions in the area.

Analysing the sensitivity of the abstraction flow rates, we observed that increased abstraction volumes would likely cause a decline in the water table with time (Fig. 7).

Mining activities will have a greater impact than agricultural activities, which means that ISL mining will inevitably have a significant impact on the water levels of the

Fig. 3 Relationship between aquifer recharge and evapotranspiration in the arid climatic condition of the STAS

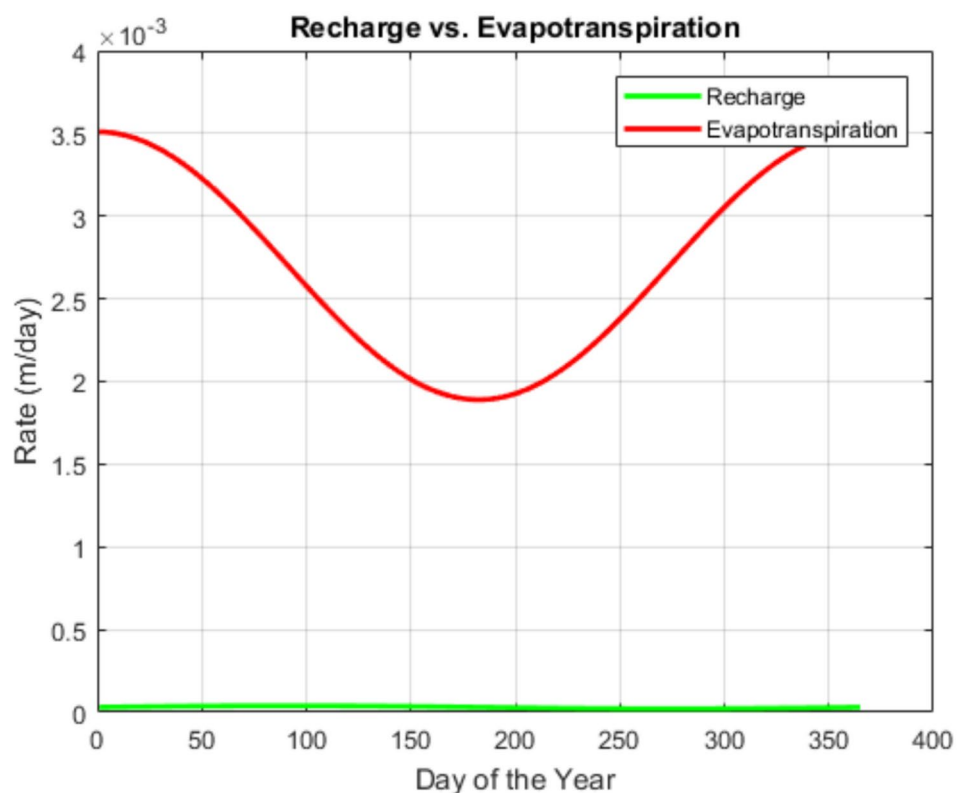


Fig. 4 Recharge rates comparisons with respect to time

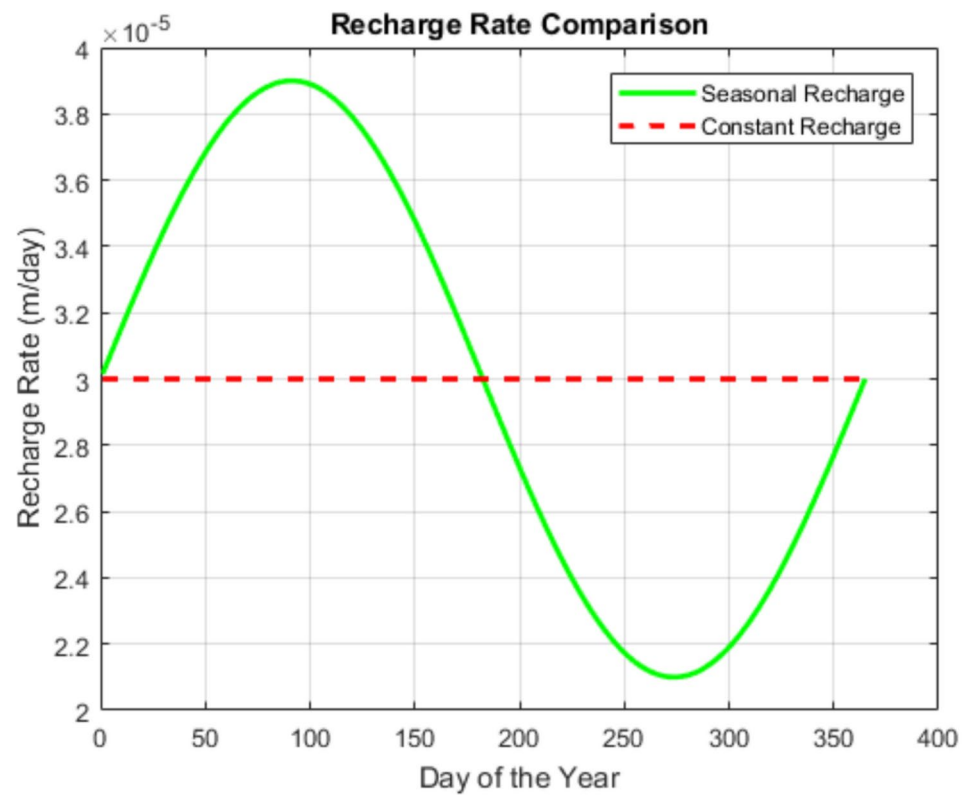


Fig. 5 Accumulation of abstraction and injection water used over 12 months

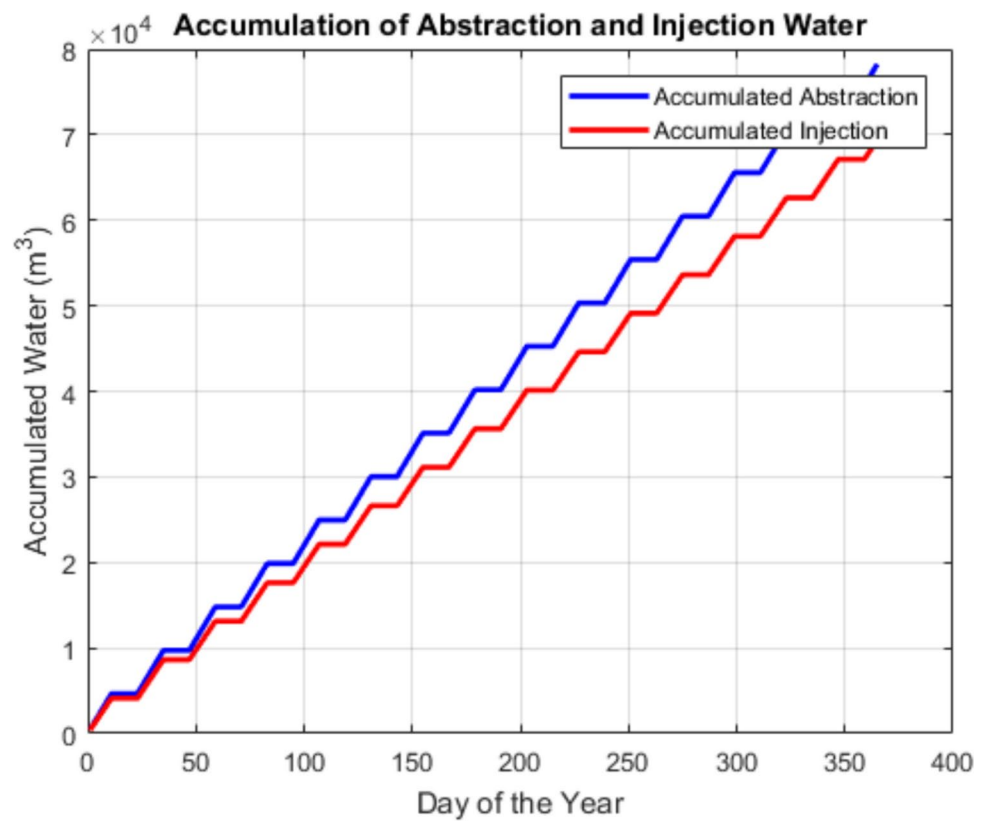


Fig. 6 The effect of injection boreholes on the water levels

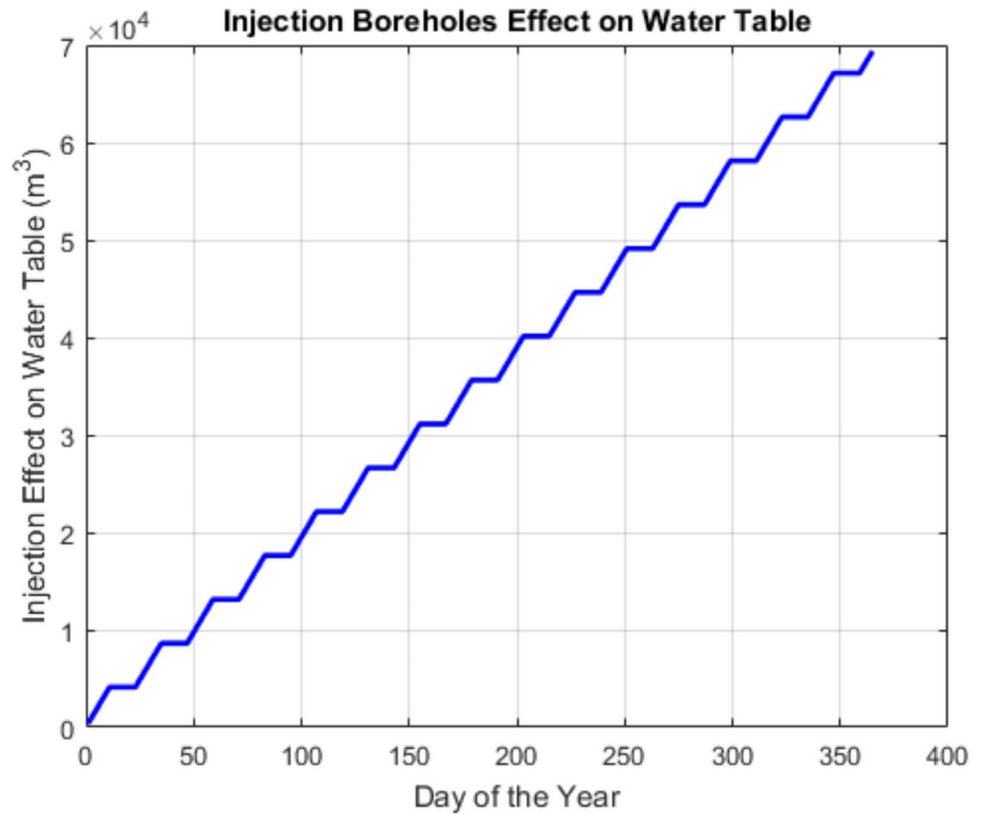
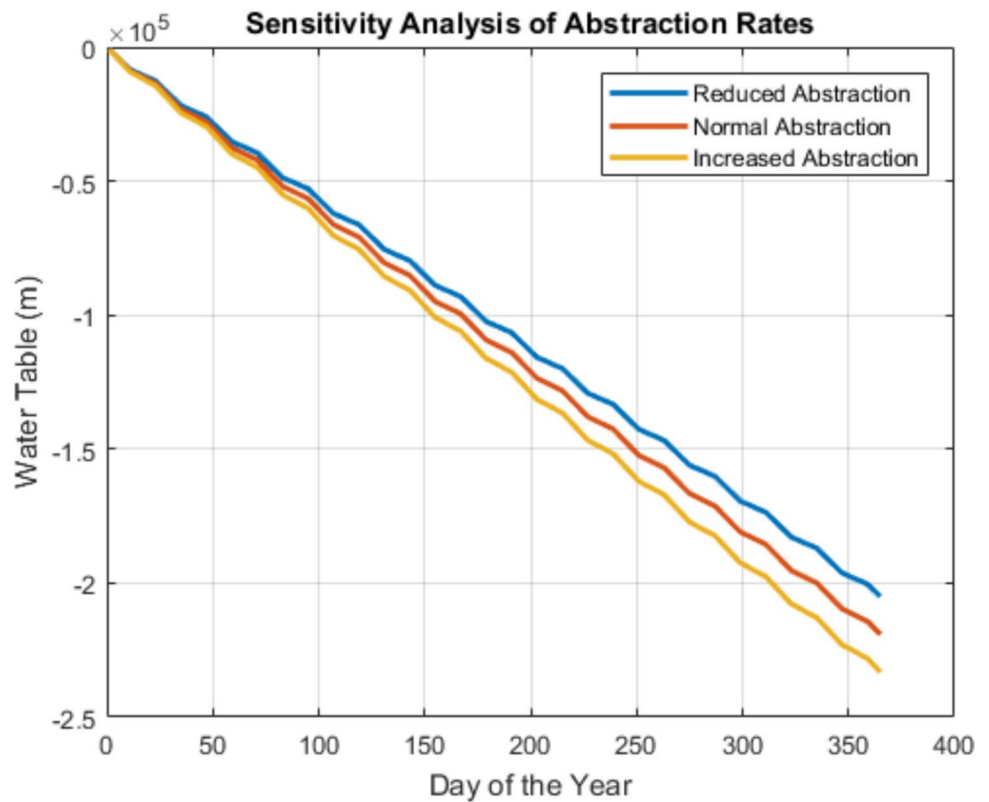


Fig. 7 Show the impact of continuous abstraction of water from the aquifer on its water level



Auob aquifer over time. The decrease in aquifer pressure would facilitate the release of sufficient water, accelerating the removal of the generated parent solution. However, the lack of actual data from the pilot testing prevented this paper from validating the extent of the impact on water levels under normal circumstances. We project that the farmers' boreholes will have a negligible impact on the pumping well of the ISL system when the radius of influence is 100 m (Fig. 8). However, the hydrogeological properties of the aquifer would simultaneously determine the extent of the influence between the irrigation and ISL boreholes.

Therefore, considering the influences of the abstraction and injection rates during ISL mining—the abstraction from farmers' boreholes as well as the effect of precipitation that may end up as groundwater recharge for this deep aquifer—we observed that overall water levels will decline with time (Fig. 9).

3.9.1 Interactions of ISL Wells and Farmers' Boreholes

We further simulated the influence of the selected flow rates of each borehole on one another, particularly with the intentional creation of the cone of depression during the ISL mining operations. The purpose of creating a deep cone of depression is mainly to control the migration of contaminants beyond the mining zone. Therefore, we numerically simulated the interaction of the boreholes with others, and the figures below illustrate the results. Figures 10 and 11

demonstrate that the irrigation boreholes' cone of depression is significantly smaller than the ISL pumping wells', as the mine site extracts more water from the aquifer than the farmers do. The farmer's borehole, with a higher production rate, obviously will create a deeper cone of depression than others.

Figures 12 and 13 indicate that the impact of the farmers' boreholes on the central borehole (ISL pumping well) would be negligible. However, if the farmer's boreholes are located near the mine site, there is a high probability that the ISL pumping well's cone of depression could interfere with them over an extended period of time.

4 Conclusions

This study extends the existing knowledge on the Auob aquifer with a detailed mathematical and numerical framework, which can be utilized for understanding and management of impacts due to in-situ leach (ISL) uranium mining in heterogeneous aquifer systems. The model clarifies the main parameters driving groundwater behaviour during in situ leaching ISL performance by employing a mix of theoretical derivations with sensitivity analyses, asymptotic behaviour assessments, and mass balance evaluations. The results highlight the significance of controlling the cone of depression and optimizing reinjection methods to contain leachate within the limits of the mining site, thus reducing

Fig. 8 Influence of farmers' boreholes on the pumping well

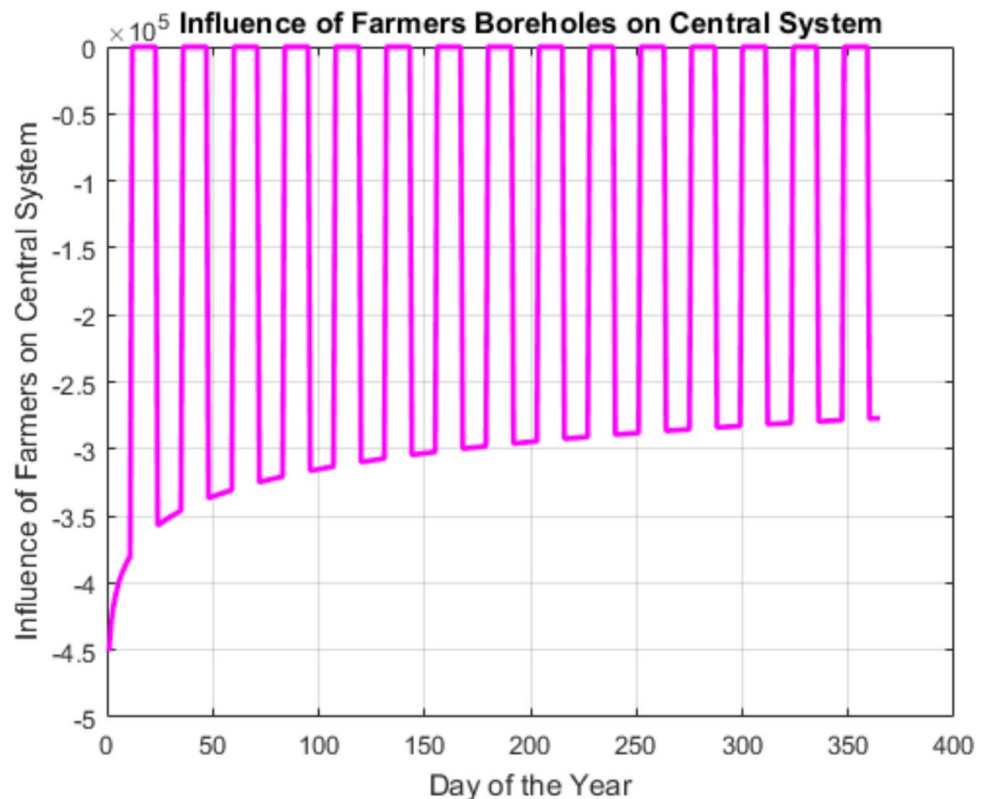


Fig. 9 Water Table fluctuations over one year

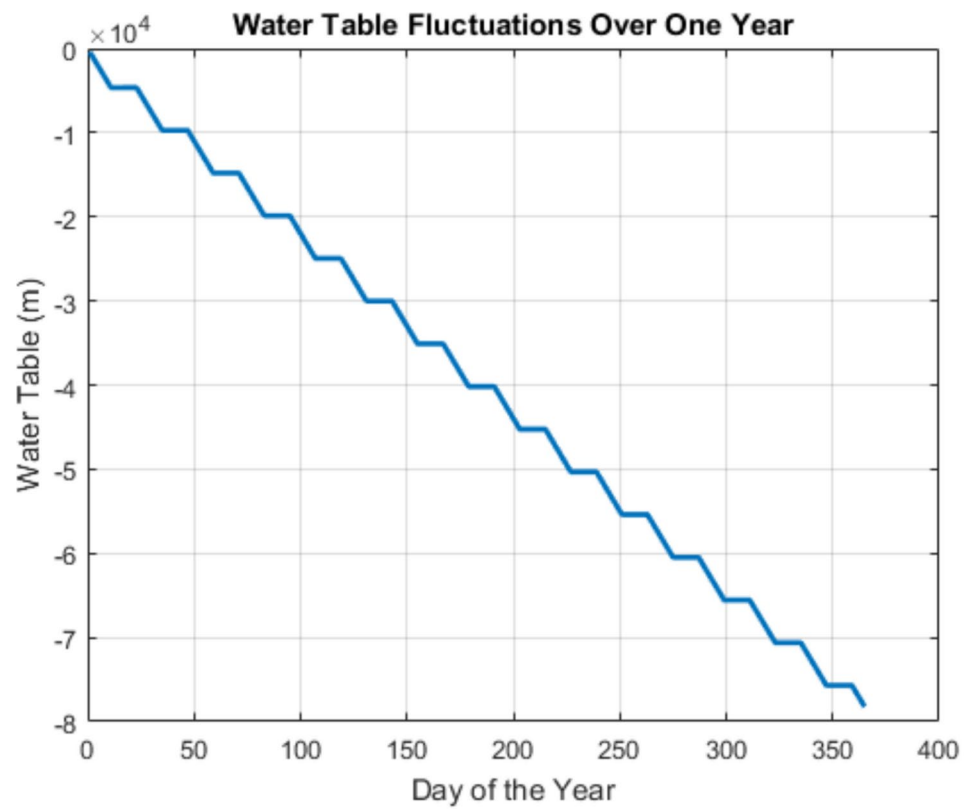


Fig. 10 Cone of depression around the central borehole (ISL pumping well)

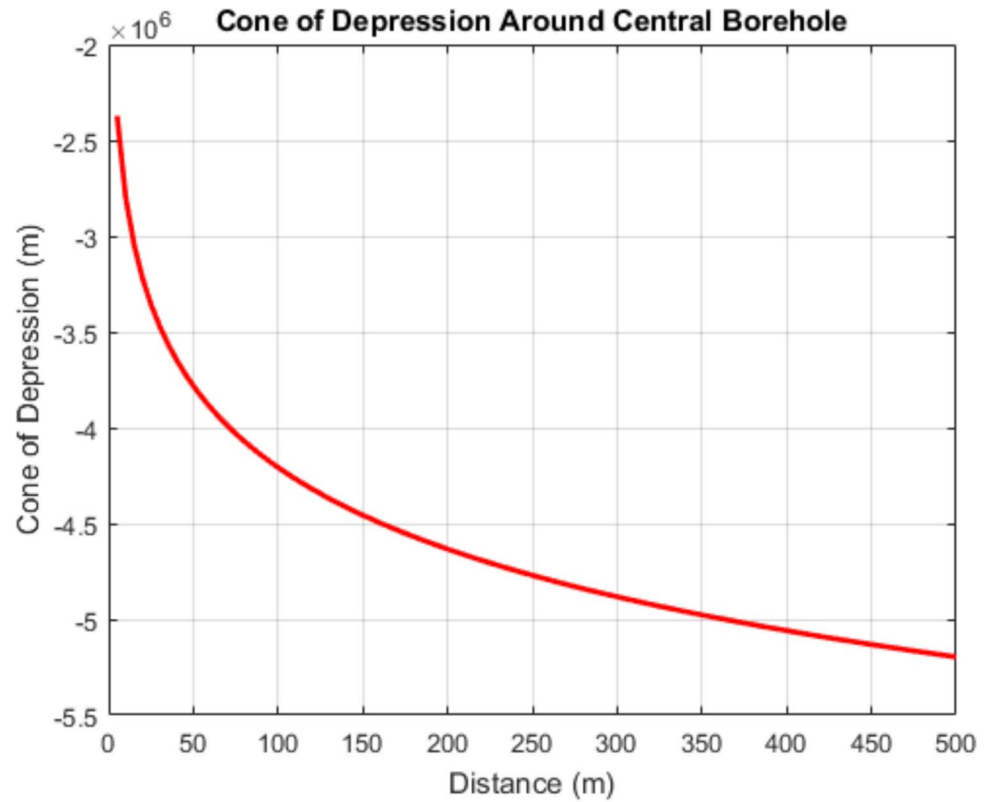


Fig. 11 Cones of depression around the farmers' boreholes

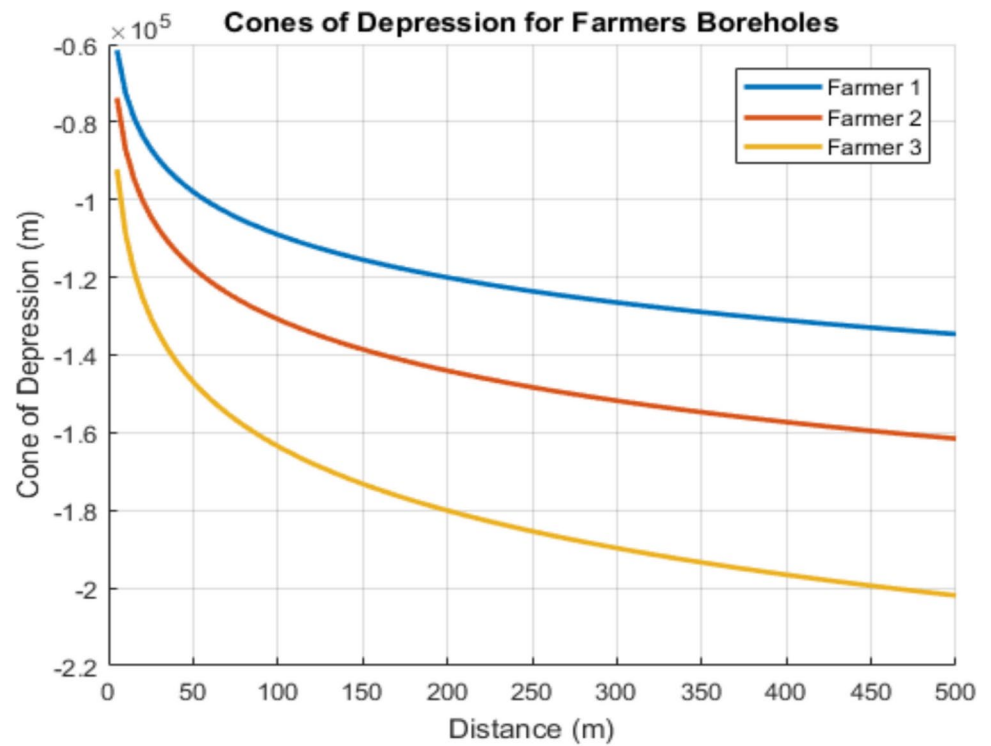


Fig. 12 simulation of the interference between the ISL central borehole and the farmers' boreholes

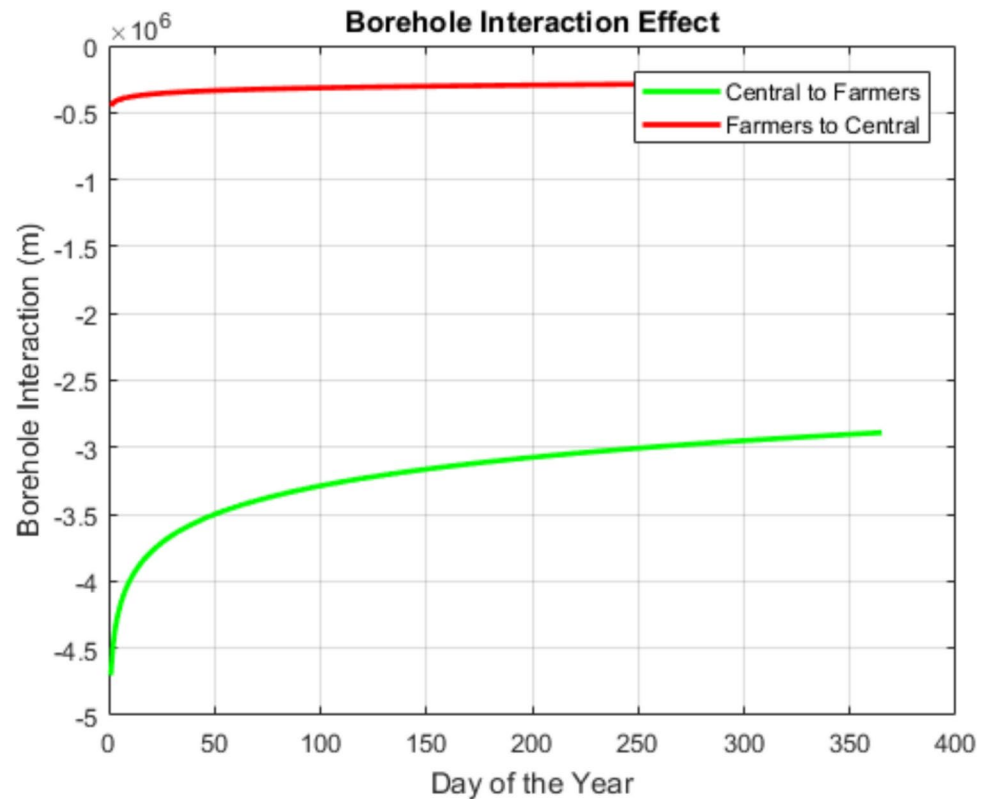
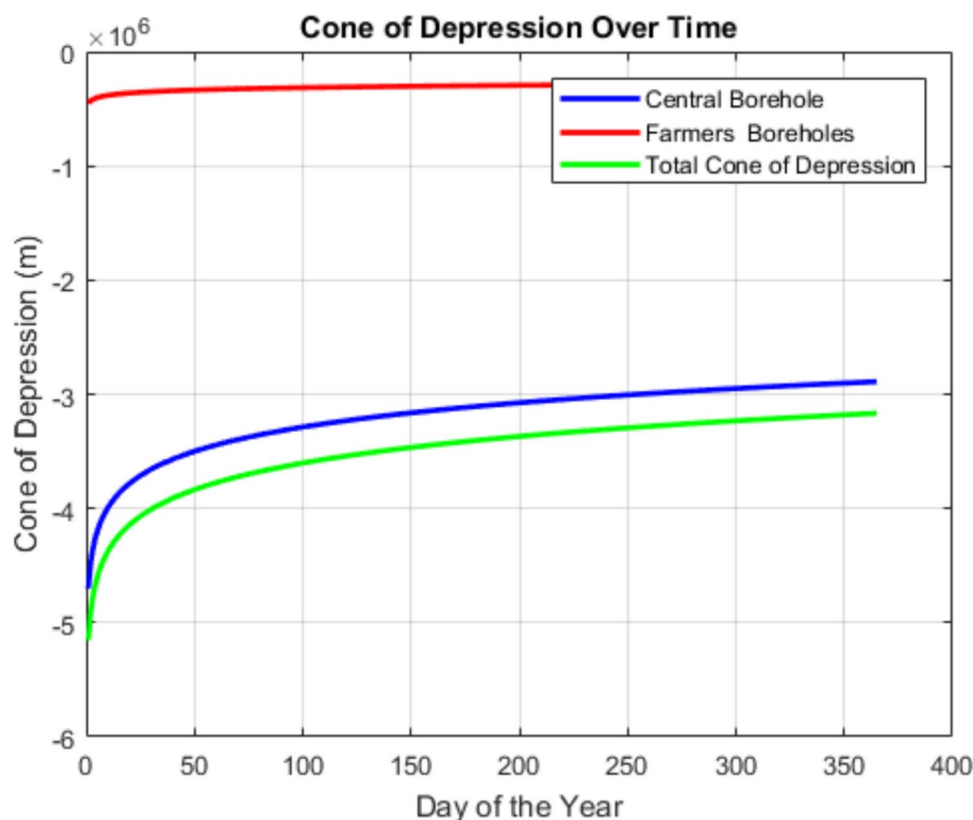


Fig. 13 Simulation of the cones of depressions generated during the ISL mining and agricultural irrigations



the negative impact on water quality and quantity. Sensitivity analysis highlights the importance of storativity and transmissivity in determining the transport of leachate and the availability of water. Additionally, the transfer efficiency effects highlight that improved containment strategies may be achievable with optimized reinjection. Such an integrated approach allows decision-makers to gain valuable insight and gives them the ability to establish a sustainable framework for the management of the Stampriet Transboundary Aquifer System in the context of ISL mining. Further studies need to include field data and extend run duration to properly parameterize the model and increase its applicability.

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Declarations

Competing interests On behalf of all authors, the corresponding author states that there is no conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

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